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OPTIMAL MANAGEMENT OF THE PARAMETERS OF A FOCUSING FIELD FOR OBTAINING A MINIMUM-EMITTANCE BEAM

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We formulate and solve the problem of optimal management for the equations of the dynamics of a Gaussian high-current relativistic electron beam in magnetic fields with a quadrupole and octupole symmetry and present the results of numerical simulation.

Introduction. The production of intense relativistic beams with a minimum transverse emittance is at present one of the basic problems in the physics of bundles of charged particles [1, 2]. Using the method of moments of a distribution function, a system of differential equations is obtained in [3] for describing the dynamics of a Gaussian high-current electron beam in magnetic fields with a quadrupole and octupole symmetry with account for the 4th-degree nonlinearity. In what follows, for this system of equations we formulated and solved the problem of optimal management of the parameters of a focusing system to obtain transport of a beam with the minimum possible transverse emittance at the outlet from a channel.

Momental Equations. We consider a relativistic beam of electrons with charge q and mass m moving along the z axis with velocity v_0 in magnetic fields with quadrupole and octupole components, for example, transport of a beam in a channel formed by quadrupole lenses with account for their edge effects (nonlinear octupole component of a focusing field). The beam is assumed to be a high-current one, but its current I does not exceed the Alfvén current $I_A = \beta\gamma mc^3/q$, where $\gamma = 1/\sqrt{1-\beta^2}$, $\beta^2 c^2 = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Therefore, the motions of particles in the longitudinal and transverse directions are decoupled, and in the linear approximation for small terms $x' = \dot{x}/\dot{z}$ and $y' = \dot{y}/\dot{z}$ (the superscript dot denotes the derivative with respect to t) for the velocity of an arbitrary particle of the beam we have $v \approx \dot{z} \approx v_0$. It is assumed that the shape of the beam in the cross section is close to elliptical, i.e., the distortions of elliptical lines of the constant level of charge density due to the nonlinearity of intrinsic and external fields are the 2nd-order effects.

The change in the transverse root-mean-square quantities $\tilde{x} = \sqrt{\overline{x^2}}$ and $\tilde{y} = \sqrt{\overline{y^2}}$ and the mean coordinates of the beam $\bar{x}$ and $\bar{y}$ along the focusing channel is described by the equations for the 2nd-order moments [3]:

$$\ddot{\tilde{x}}\tilde{x} + \dot{\tilde{x}}\dot{\tilde{x}} = \frac{1}{\gamma^2 m^2}\rho_x - \frac{eS^2}{\gamma m\beta c}\left(g\tilde{x}^2 - \frac{1}{12}g''\tilde{x}^4 - \frac{1}{4}g''\overline{x^2y^2}\right) + \frac{eS^2}{\gamma^3 m\beta^2 c^2}\left(C_1\tilde{x}^2 + C_2\tilde{x}^4 + C_3\overline{x^2y^2}\right),$$

$$\ddot{\tilde{y}}\tilde{y} + \dot{\tilde{y}}\dot{\tilde{y}} = \frac{1}{\gamma^2 m^2}\rho_y + \frac{eS^2}{\gamma m\beta c}\left(g\tilde{y}^2 - \frac{1}{12}g''\tilde{y}^4 - \frac{1}{4}g''\overline{x^2y^2}\right) + \frac{eS^2}{\gamma^3 m\beta^2 c^2}\left(C_4\tilde{y}^2 + C_5\tilde{y}^4 + C_6\overline{x^2y^2}\right),$$

$$\dot{\rho}_x = -2ev_0\gamma m\left(g\tilde{x}\dot{\tilde{x}} - \frac{1}{12}g''\tilde{x}^3\dot{\tilde{x}} - \frac{1}{4}g''\overline{xy^2\dot{x}}\right) + \frac{2e\gamma m}{\gamma^2}\left(C_1\tilde{x}\dot{\tilde{x}} + C_2\tilde{x}^3\dot{\tilde{x}} + C_3\overline{xy^2\dot{x}}\right),$$

$$\dot{\rho}_y = +2ev_0\gamma m\left(g\tilde{y}\dot{\tilde{y}} - \frac{1}{12}g''\tilde{y}^3\dot{\tilde{y}} - \frac{1}{4}g''\overline{yx^2\dot{y}}\right) + \frac{2e\gamma m}{\gamma^2}\left(C_4\tilde{y}\dot{\tilde{y}} + C_5\tilde{y}^3\dot{\tilde{y}} + C_6\overline{yx^2\dot{y}}\right), \quad (1)$$

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$$\ddot{\bar{x}} = -\frac{eS^2}{\gamma m\beta c}\left(g\bar{x} - \frac{1}{12}g''\overline{x^3} - \frac{1}{4}g''\overline{xy^2}\right) + \frac{eS^2}{\gamma^3 m\beta^2 c^2}\left(C_1\bar{x} + C_2\overline{x^3} + C_3\overline{xy^2}\right),$$

$$\ddot{\bar{y}} = +\frac{eS^2}{\gamma m\beta c}\left(g\bar{y} - \frac{1}{12}g''\overline{y^3} - \frac{1}{4}g''\overline{x^2y}\right) + \frac{eS^2}{\gamma^3 m\beta^2 c^2}\left(C_1\bar{y} + C_2\overline{y^3} + C_3\overline{x^2y}\right),$$

where $\rho_x = \overline{\dot{x}^2}$, $\rho_y = \overline{\dot{y}^2}$. The dot denotes the derivative with respect to the independent variable τ: $\tau \in [\tau_0; \tau_0 + T]$, where $T = P/S$, S is the period of the focusing structure; P is the transport length, $d\tau = dz/S = (v_0/S)dt$. The gradient of a separate quadrupole and the second derivative (with respect to z) of the gradient are denoted by g and g'', respectively.

The 4th-order moments $\overline{x^2y^2}$, $\overline{x^2y\dot{y}}$, $\overline{y^2x\dot{x}}$ are determined from the system of equations

$$\begin{aligned}
\dot{b}_1 &= b_3 + a_y b_7 + 2b_8, & \dot{b}_4 &= 2a_x b_2 + 2b_5, & \dot{b}_7 &= 2b_1 + 2b_2,\\
\dot{b}_2 &= b_4 + a_x b_7 + 2b_8, & \dot{b}_5 &= a_y b_4 + 2a_x b_8 + b_9, & \dot{b}_8 &= a_x b_1 + a_y b_2 + b_5 + b_6,\\
\dot{b}_3 &= 2a_y b_1 + 2b_6, & \dot{b}_6 &= a_x b_3 + 2a_y b_8 + b_9, & \dot{b}_9 &= 2a_x b_5 + 2a_y b_6,
\end{aligned} \tag{2}$$

where

$$a_x(z) = -\frac{eS^2}{\gamma m\beta c}g(z) + \frac{eS^2}{\gamma^3 m\beta^2 c^2}C_1(z); \quad a_y(z) = \frac{eS^2}{\gamma m\beta c}g(z) + \frac{eS^2}{\gamma^3 m\beta^2 c^2}C_4(z);$$

$$b_1 = \overline{x^2 y\dot{y}};\ b_2 = \overline{y^2 x\dot{x}};\ b_3 = \overline{x^2\dot{y}^2};\ b_4 = \overline{\dot{x}^2 y^2};\ b_5 = \overline{\dot{x}^2 y\dot{y}};\ b_6 = \overline{\dot{y}^2 x\dot{x}};$$

$$b_7 = \overline{x^2y^2};\ b_8 = \overline{x\dot{x}y\dot{y}};\ b_9 = \overline{\dot{x}^2\dot{y}^2}.$$

The 3rd-order moments $\overline{xy^2}$, $\overline{yx^2}$, $\overline{x^3}$, $\overline{y^3}$ are determined from the expressions

$$\begin{gathered}
\overline{x^3} = 3\bar{x}\overline{x^2} - 2\bar{x}^3, \quad \overline{y^3} = 3\bar{y}\overline{y^2} - 2\bar{y}^3,\\
\overline{x^2y} = -2\bar{x}^2\bar{y} + 2\bar{x}\,\overline{xy} + \overline{x^2}\,\bar{y}, \quad \overline{y^2x} = -2\bar{y}^2\bar{x} + 2\bar{y}\,\overline{xy} + \overline{y^2}\,\bar{x},
\end{gathered} \tag{3}$$

where $\overline{xy} = ((\overline{x^2y^2} - \bar{x}^2\bar{y}^2 + 2\bar{x}^2\bar{y}^2)/2)^{1/2}$.

The coefficients C_i ($i = 1, \ldots, 6$) are constant along the step l of numerical integration of system (1); they are connected with the coefficients of polynomial expansion of the space-charge field potential

$$\varphi^{\text{beam}}(x, y, z) = C_{01}(z)x^2 + C_{02}(z)y^2 + C_{40}(z)x^4 + C_{04}(z)y^4 + C_{22}(z)x^2y.$$

by the relationships $C_1 = -C_{20}$, $C_2 = -C_{40}$, $C_4 = -C_{02}$, $C_5 = -C_{04}$, $C_3 = C_6 = -C_{22}$.

Optimization of the Focusing-System Parameters. The problem of optimal management is prescribed by the equations of state and phase limitations that determine the behavior of the object, as well as by the quality criterion [4]. In the case considered, the equations of state involve a system of 2nd-order differential equations (1) together with a system of 1st-order differential equations (2) and expressions (3). The phase limitations determine the region of the change in unknown functions, independent variable, and managing parameters. The latter are considered to be the values of the quadrupole lens gradient and of its second derivative.

Let us formulate the problem of optimal management: *determine the parameters of quadrupoles (the gradient of a lens, the second derivative of the gradient) as functions of the longitudinal coordinate z that ensure the production of a beam with the minimum possible transverse emittance.*

We will rewrite the system of equations (1)-(2) in the form

$$\frac{dZ}{d\tau} = f(\tau, Z(\tau, u), u(\tau)), \quad 0 \le \tau \le T, \tag{4}$$

where

$$Z(\tau, u) = \{\tilde{x}, \dot{\tilde{x}}, \tilde{y}, \dot{\tilde{y}}, \rho_x, \rho_y, \bar{x}, \dot{\bar{x}}\, \bar{y}, \dot{\bar{y}}, b_1, b_2, b_3, b_4, b_5, b_6, b_7\, b_8, b_9\}; \; u(\tau) = \{g(\tau, g''(\tau)\}.$$

With $\tau = 0$ and prescribed management $u_0(\tau)$, the initial condition for (4) is assumed in the form $Z(0, u_0(0)) = Z_0$. First, we will not impose any limitations on the values of phase changes for $0 < \tau \le T$. The limitation on the management $u(\tau)$ is presented as $\alpha_1 \le g(z) \le \beta_1$, $\alpha_2 \le g''(z) \le \beta_2$, where α_1, α_2 determine the lower boundaries of the region of change in the management and β_1, β_2 determine the upper boundaries.

We will consider the problem of determining the management $u(t)$ and trajectory $Z(t)$ that deliver the minimum to the functional $\Phi(Z(t, u)) = \Phi(t, Z(t, u), u(t))$ at $t = T$ (Mayer's problem). As a minimizing functional, we will consider

$$\Phi(Z(t, u)) = \tilde{x}^2 \rho_x - \tilde{x}^2 \dot{\tilde{x}}^2 + \tilde{y}^2 \rho_y - \tilde{y}^2 \dot{\tilde{y}}^2. \tag{5}$$

To solve the problem set, we will use Pontryagin's maximum principle [5], which makes it possible to reduce the problem of optimal management to a special boundary-value problem for a system of ordinary differential equations.

It is known that for problems with a free end there is generally no procedure for obtaining an exact solution. However, if the initial system of equations is linear in phase variables and no limitations are imposed on the right end of the trajectory, then in such a case the problem of optimal management with a free end is reduced to the solution of two Cauchy problems for conjugated and initial systems of equations with account for the condition of the maximum for the Hamilton function [5].

To solve the above-formulated nonlinear terminal (with a fixed time) problem with a fixed left end and free right end, we shall make use of the Brison–Shatrovsky method [4]. This iterative method at each step operates with "dispatching management" $u_*(t)$ and with the corresponding "dispatcher solution" $Z_*(t)$ of the equations of state. The convergence of this method is ensured by the concavity of the Hamilton function toward $u(t)$ and the convexity of the multitude of admissible managements U [6] which is valid for the problem formulated. Then, for any initial admissible management $u_*(t)$ and corresponding solution $Z_*(t)$, this method determines a sequence of managements $u_k \in U$ such that $\| u_k - u_0 \| \to 0$, when $k \to \infty$ in the norm of the space $L_2(0, T)$ of the functions integrable with a square, where u_0 is the desired management that satisfies the maximum principle.

We shall describe the step of iteration of the method for system (4). Using the new variables $Z_1 = Z - Z_*$ and $u_1 = u - u_*$, we linearize the initial system of equations (4) at fixed values of Z_*, u_* and write for it the following Cauchy problem:

$$\dot{Z}_1 = A(t) Z_1 + B(t) u_1, \tag{6}$$

$$Z_1(0) = 0. \tag{7}$$

We calculate the quality criterion gradient and write the Cauchy problem for a conjugated system:

$$\frac{\partial \psi}{\partial t} = - A(t)^T \psi, \tag{8}$$

$$\psi(T) = - \operatorname{grad} \Phi(Z(T, u_*)|_{Z=Z_*}. \tag{9}$$

The Hamilton function for the linearized problem can be represented in the form of a scalar product of the functions $Z_1(t, u_1)$ and $\psi(t)$:

$$H(t, Z_1(t), \psi(t), u_1(t)) = (\psi, AZ_1) + (\psi, Bu_1).$$

The maximum principle $\forall\, t \in [t_0, T]$ has the form

$$(\psi(t), B(t)\, u_1(t)) = \max_{u_* + u_1 \in U} (\psi(t), B(t)\, u_1(t)). \tag{10}$$

For the value of u_1 selected, a decrease in the quality criterion $J = \Phi(Z_1(T, u_1))$ occurs on satisfaction of the inequality

$$\delta J < 0. \tag{11}$$

Since the problem of determining the minimum of the functional $\Phi(Z(T, u))$ is not identical to the problem of minimization of the functional J, together with condition (11) it is necessary also to check the condition

$$\Phi(Z(T, u) < \Phi(Z_*(T, u_*)). \tag{12}$$

By virtue of the fact that for the physical system considered (a beam in focusing magnetic fields) there are actual limitations on the coordinates and velocities of the SREP particles (along the entire transport channel the beam radius must not exceed the transverse dimension of the vacuum tube r_0, whereas the absolute values of the transverse velocities of particles are always smaller than v_0), within the framework of the above-formulated problem of optimal management, phase limitations appear that are imposed on system (4). The removal of the indicated limitations will be made by the method of penalty functions [4]. We will modernize the quality criterion by having incorporated into it the integral penalty term:

$$\Phi_1(Z(T, u)) = \Phi(Z(T, u)) + \int_0^{T_*} dt\, (\lambda_1 [(x_b - \tilde{x})^2 \theta(\tilde{x} - x_b) + (y_b - \tilde{y})^2 \theta(\tilde{y} - y_b)] +$$

$$+ \lambda_2 [(\nu_b - \rho_x)^2 \theta(\rho_x - \nu_b) + (\mu_b - \rho_y)^2 \theta(\rho_y - \mu_b)]), \tag{5'}$$

where $\theta(x)$ is the Heaviside theta function; $x_b = r_0$, $y_b = r_0$, $\nu_b = v_0$, $\mu_b = v_0$. At the beginning of the process of iteration, the penalty coefficients $\lambda_1, \lambda_2 << 1$. They increase in the course of iterations until $\tilde{x}, \tilde{y}, \rho_x$, and ρ_y completely satisfy the limiting conditions or until we see that solution of the problem does not exist under the given limitations.

An algorithm for computing the solution of the formulated problem of optimal management can be described as follows:

(a) fix the "dispatching management" and find the "dispatcher solution" of the initial system (4);

(b) write the linearized system (6)-(7);

(c) solve the Cauchy problem (8)-(9) from right to left for $\psi(t)$;

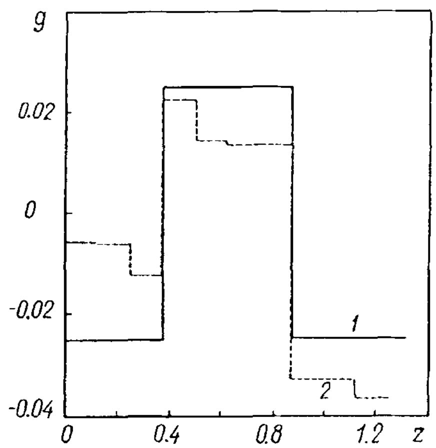
(d) according to the maximum principle (10), select the variation of the management δu and find the new magnitude of the management $u^{(1)} = u_* + \delta u$;

(e) integrate system (4) from left to right and determine $Z(t, u^{(1)}(t))$;

(f) find the value of the functional $\Phi(Z(T, u^{(1)}(t)))$ according to (5') on a certain solution $Z(t, u^{(1)}(t))$ corresponding to the selected management $u^{(1)}$ and check condition (12).

If condition (12) is satisfied, the quality criterion is decreased. In such a case, we take the management found as a new "dispatching management" and repeat the above procedure beginning from the first step. If condition (12) is not satisfied, we gradually decrease the variation of the management by a factor of two and check the vectors $u^{(2)} = u_* + (1/2)\delta u$, $u^{(3)} = u_* + (1/4)\delta u$, etc., repeating the above procedure, starting from the fourth step.

Thus, the search for the solution of the formulated nonlinear terminal problem incorporates global management iterations, associated with finding a series of "dispatcher solutions," and local iterations on the variation of management as a version of descent in minimization of δJ.



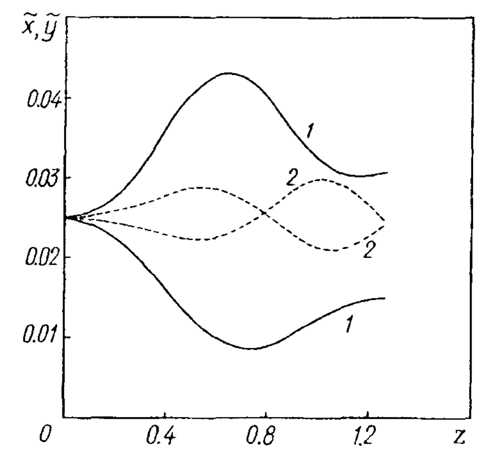


Fig. 1. Disposition and magnitude of the gradients of quadrupole lenses along the focusing channel: 1, 2) initial and optimal parameters of the focusing system. z, m; g, T/m.

Fig. 2. Root-mean-square dimensions of the beam transported: 1, 2) initial and optimal parameters of the focusing system. $\tilde{x}$, $\tilde{y}$, m.

The iteration process is repeated until either the antigradient of the quality criterion, calculated on the right end, becomes positive (in this case, the optimality of the solution obtained in the last iteration follows from the assumption of the uniqueness of the solution of the problem [4] and of the convergence of the method proved in [6]) or simultaneously the following two conditions are not fulfilled:

1. The relative error of the quality criterion calculated in successive iterations is smaller than the prescribed epsilon 1 (the optimum solution is found within the prescribed accuracy).

2. The absolute value of the management variation became smaller than epsilon 2 (successive values of managements differ within the limits of the prescribed accuracy and further optimization leads to a change in the quality criterion within the error of calculation).

Integration of the system of differential equations in each iteration is performed numerically by the Runge–Kutta 4th-order method. The solution of the problem of optimal management by the parameters of a focusing field is implemented in the SREP program.

Results of Optimal Management. As an example, we shall consider the transport, by quadrupole multiplets, of an electron beam with the following parameters: the current $I = 1$ kA, the energy of particles $E = 1$ MeV, $\tilde{x}(0) = \tilde{y}(0) = 2.5$ cm, $\dot{\tilde{x}}(0) = \dot{\tilde{y}}(0) = 0$, the scatter over transverse velocities amounts to 1% of the value of the longitudinal velocity.

The graphs of the prescribed initial and obtained optimal values of the gradients of quadrupole lenses g are presented in Fig. 1. In the course of optimization, the value of g'', which characterizes the nonlinearity of the focusing field, turns out to be identical for all the sections of the transport channel and equal to -0.2 T/m^3. This seems to be associated with the fact that for this example the quadratic terms in the expansion $\varphi^{\text{beam}}(x, y, z)$ considerably exceed the 4th-order terms and a partial compensation of the increase in the beam emittance due to the nonlinearities of the field of the space charge is possible by the constant function g''. The value of the functional (5) for the beam at the outlet of the transport channel in the case of the prescribed initial management is equal to $J = 1.875 \cdot 10^{-7}$ (m·rad)^2; the optimal configuration of the focusing fields gives $J^{\text{opt}} = 1.874 \cdot 10^{-7}$ (m·rad)^2. To obtain this result, 74 global iterations were carried out by the SREP program.

The graphs of the change in the mean-square dimensions of the beam along the transport channel that correspond to the prescribed and optimal parameters of quadrupoles are presented in Fig. 2. It is seen from the graphs that owing to optimization by transporting in the presence of nonlinearities of focusing and space-charge fields, it is possible to decrease the radius of the beam without increasing its transverse emittance.

The correctness of the above-formulated problem of optimal management confirms the following fact. When, for the example considered, optimal management was selected as a prescribed initial management (g is prescribed in the form of a piecewise-constant function according to curve 2 of Fig. 1 and $g'' = -0.2$ T/m^3), the SREP program performed a single global iteration and the prescribed management is issued as optimal management. And here again $J^{\mathrm{opt}} = 1.874 \cdot 10^{-7}$ (m·rad)^2.

Conclusion. We note that, generally speaking, the obtained solution of the above-stated nonlinear problem of optimal management will not satisfy the maximum principle, since with the use of gradient methods the approximation to the optimal management breaks down in the class of relay (boundary) managements [7]. However, owing to the convergence of the iteration process [6], it is possible to attain a situation where the difference between the optimum value of the quality criterion (5) and that obtained by the method indicated will be arbitrarily small.

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